

Heisenberg Double and Pentagon Relation

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Abstract

It is shown that the Heisenberg double has a canonical element, satisfying the pentagon relation. From a given invertible constant solution to the pentagon relation one can restore the structure of the underlying algebras. Drinfeld double can be realized as a subalgebra in the tensor square of the Heisenberg double. This enables one to write down solutions to the Yang-Baxter relation in terms of solutions to the pentagon relation.

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1 Introduction

The theory of quantum groups [3] appeared as an algebraic setting for the construction of solutions to the Yang-Baxter equation [12, 2]:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (1.1)$$

According to [3], for any Hopf algebra \mathcal{A} one can construct a quasi-triangular Hopf algebra $D(\mathcal{A})$, called Drinfeld double, where the universal R -matrix, satisfying relation (1.1), exists. In [5] it was shown how to reconstruct the underlying Hopf algebra from a given solution to equation (1.1). In [9, 1, 10] another, Heisenberg double $H(\mathcal{A})$, has been introduced, which, unlike the Drinfeld double, is not a Hopf algebra.

The purpose of the present letter is to show that the constant “pentagon” relation

$$S_{12}S_{13}S_{23} = S_{23}S_{12}, \quad (1.2)$$

plays the same role in Heisenberg double as Yang-Baxter relation (1.1) does in Drinfeld’s one. This is the content of Sect. 2. In Sect. 3 we show that the Drinfeld double can be realized as a subalgebra in the tensor square of the Heisenberg double. This gives an explicit formula for solutions to the Yang-Baxter equation in terms of solutions to the pentagon equation. As an example, in Sect. 4 the Borel subalgebra of $U_q(sl(2))$ at $|q| < 1$ is considered and a slightly generalized form of the quantum dilogarithm identity of [4] is obtained.

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2 Heisenberg Double and Pentagon Relation

Let $\{e_\alpha\}$ be a linear basis of associative and co-associative bialgebra \mathcal{A} , with the following multiplication and co-multiplication rules:

$$e_\alpha e_\beta = m_{\alpha\beta}^\gamma e_\gamma, \quad \Delta(e_\alpha) = \mu_\alpha^{\beta\gamma} e_\beta \otimes e_\gamma, \quad (2.1)$$

where summations over the repeated indices are implied. The Heisenberg double $H(\mathcal{A})$ can be defined as an associative algebra with generating elements $\{e^\beta, e_\alpha\}$ and the following defining relations:

$$e_\alpha e_\beta = m_{\alpha\beta}^\gamma e_\gamma, \quad e^\alpha e^\beta = \mu_\gamma^{\alpha\beta} e^\gamma, \quad e_\alpha e^\beta = m_{\rho\gamma}^\beta \mu_\alpha^{\gamma\sigma} e^\rho e_\sigma. \quad (2.2)$$

There are two subalgebras of $H(\mathcal{A})$ with the linear bases $\{e_\alpha\}$ and $\{e^\alpha\}$, which are equivalent to the algebra \mathcal{A} and its dual \mathcal{A}^* . The analog of the adjoint representation is given by a realization in terms of the structure constants:

$$\langle \alpha | e_\beta | \gamma \rangle = m_{\alpha\beta}^\gamma, \quad \langle \alpha | e^\beta | \gamma \rangle = \mu_\alpha^{\beta\gamma}. \quad (2.3)$$

The remarkable property of the Heisenberg double is described by the following theorem.

Theorem 1 *The canonical element $S = e_\alpha \otimes e^\alpha$ in the Heisenberg double satisfies the pentagon relation (1.2).*

Now, following [5], for a given invertible solution S of the pentagon relation (1.2), define two bialgebras \mathcal{B} and \mathcal{B}^* , generated with entries of matrices F and G , respectively, and the following multiplication and co-multiplication rules:

$$F_1 F_2 S_{12} = S_{12} F_1, \quad S_{12} G_1 G_2 = G_2 S_{12}, \quad (2.4)$$

$$\Delta(F_1) = F_1 \otimes F_1, \quad \Delta(G_1) = G_1 \otimes G_1. \quad (2.5)$$

Note, that the associativity conditions are ensured by the pentagon relation (1.2). In fact, these algebras are dual as bialgebras with the following pairing:

$$\langle G_1, F_2 \rangle = S_{12}. \quad (2.6)$$

To extract the structure constants for given linear bases $\{e_\alpha\}$ in \mathcal{B} and $\{e^\alpha\}$ in \mathcal{B}^* with the canonical pairing:

$$\langle e^\beta, e_\alpha \rangle = \delta_\alpha^\beta, \quad (2.7)$$

represent matrices F and G as linear combinations of the base elements with some coefficients:

$$F_1 = F_1^\alpha e_\alpha, \quad G_1 = G_{1,\alpha} e^\alpha. \quad (2.8)$$

Substituting these expressions into (2.6), and using (2.7), we obtain:

$$S_{12} = G_{1,\alpha} F_2^\alpha. \quad (2.9)$$

Expanding the known matrix S_{12} in the form of (2.9), one can calculate the matrices $G_{1,\alpha}$ and F_1^α up to an invertible transformation over the index α . From (2.9) it follows also, that for a finite dimensional matrix S one associates in this way a finite dimensional algebra with the dimension, given by the formula:

$$\dim(\mathcal{B}) = \text{rank}(P_{12} S_{12})^{t_1}, \quad (2.10)$$

where P_{12} is the permutation matrix, and the upper index t_1 means the transposition in the first subspace. Introduce now the dual matrices $F_{1,\alpha}$ and G_1^α through the equations:

$$\text{tr}_1(F_{1,\alpha} F_1^\beta) = \delta_\alpha^\beta, \quad \text{tr}_1(G_{1,\alpha} G_1^\beta) = \delta_\alpha^\beta. \quad (2.11)$$

Using these, one can derive the following formulas for the structure constants:

$$m_{\alpha\beta}^\gamma = \text{tr}_1(G_{1,\alpha} G_{1,\beta} G_1^\gamma), \quad \mu_\gamma^{\alpha\beta} = \text{tr}_1(F_1^\alpha F_1^\beta F_{1,\gamma}). \quad (2.12)$$

There is an analog of the adjoint representation for these algebras, given by the S -matrix itself:

$$F_1 = S_{01}, \quad G_1 = S_{10}, \quad (2.13)$$

where the 0-th subspace corresponds to the representation space. In fact, formulas (2.13) realize the Heisenberg double, defined by (2.4) and the mixed permutation relation of the form:

$$G_1 S_{12} F_2 = F_2 G_1. \quad (2.14)$$

Formulas (2.10)–(2.12) make sense for finite dimensional case, while the infinite dimensional case needs a further qualification. Our result can be stated as the following theorem.

Theorem 2 *For any invertible solution to the constant pentagon relation (1.2) one can associate a pair of mutually dual associative and co-associative bialgebras.*

3 Yang-Baxter and Pentagon Relations

Let $H(\mathcal{A})$ be the Heisenberg double, defined in Section 2. Co-multiplications of subalgebras \mathcal{A} and \mathcal{A}^* can not be extended to any co-multiplication of the whole algebra. This is the main difference between Heisenberg and Drinfeld doubles. It is possible, however, to realize Drinfeld double as a subalgebra in the tensor product of two Heisenberg's, $H(\mathcal{A}) \otimes \tilde{H}(\mathcal{A})$, where the second “tilded” double is defined as follows:

$$\tilde{e}_\alpha \tilde{e}_\beta = m_{\alpha\beta}^\gamma \tilde{e}_\gamma, \quad \tilde{e}^\alpha \tilde{e}^\beta = \mu_\gamma^{\alpha\beta} \tilde{e}^\gamma, \quad \tilde{e}^\beta \tilde{e}_\alpha = \mu_\alpha^{\sigma\gamma} m_{\gamma\rho}^\beta \tilde{e}_\sigma \tilde{e}^\rho. \quad (3.1)$$

The canonical element $\tilde{S} = \tilde{e}_\alpha \otimes \tilde{e}^\alpha$, satisfies the “reversed” pentagon relation:

$$\tilde{S}_{12} \tilde{S}_{23} = \tilde{S}_{23} \tilde{S}_{13} \tilde{S}_{12}. \quad (3.2)$$

Using relations (2.2) and (3.1) it is easy to show that the elements

$$E_\alpha = \mu_\alpha^{\beta\gamma} e_\beta \otimes \tilde{e}_\gamma, \quad E^\alpha = m_{\gamma\beta}^\alpha e^\beta \otimes \tilde{e}^\gamma. \quad (3.3)$$

satisfy the defining relations of the Drinfeld double:

$$E_\alpha E_\beta = m_{\alpha\beta}^\gamma E_\gamma, \quad E^\alpha E^\beta = \mu_\gamma^{\alpha\beta} E^\gamma, \quad \mu_\alpha^{\sigma\gamma} m_{\gamma\rho}^\beta E_\sigma E^\rho = m_{\rho\gamma}^\beta \mu_\alpha^{\gamma\sigma} E^\rho E_\sigma. \quad (3.4)$$

In particular, for the canonical element $R = E_\alpha \otimes E^\alpha$ one gets a factorized formula:

$$R_{12,34} = S'_{14} S_{13} \tilde{S}_{24} S'_{23}, \quad (3.5)$$

where

$$S' = \tilde{e}_\alpha \otimes e^\alpha, \quad S'' = e_\alpha \otimes \tilde{e}^\alpha. \quad (3.6)$$

By construction, R -matrix (3.5) satisfies the Yang-Baxter relation:

$$R_{1\bar{1},2\bar{2}} R_{1\bar{1},3\bar{3}} R_{2\bar{2},3\bar{3}} = R_{2\bar{2},3\bar{3}} R_{1\bar{1},3\bar{3}} R_{1\bar{1},2\bar{2}}. \quad (3.7)$$

In fact, it is a consequence of eight different pentagon relations, two homogeneous ones, (1.2) and (3.2), and six mixed ones:

$$\begin{aligned} S'_{12} S'_{13} S_{23} &= S_{23} S'_{12}, & \tilde{S}_{12} S'_{23} &= S'_{23} S'_{13} \tilde{S}_{12}, \\ S_{12} S''_{13} S''_{23} &= S''_{23} S_{12}, & S''_{12} \tilde{S}_{23} &= \tilde{S}_{23} S''_{13} S''_{12}, \\ S'_{12} \tilde{S}_{13} S''_{23} &= S''_{23} S'_{12}, & S''_{12} S'_{23} &= S'_{23} S_{13} S''_{12}. \end{aligned} \quad (3.8)$$

Consider now the case, where algebra \mathcal{A} is a Hopf algebra. The unit element and co-unity map have the form:

$$1 = \varepsilon^\alpha e_\alpha, \quad \epsilon(e_\alpha) = \varepsilon_\alpha, \quad (3.9)$$

and there are also the antipode and it's inverse maps:

$$\gamma(e_\alpha) = \gamma_\alpha^\beta e_\beta, \quad \bar{\gamma}(e_\alpha) = \bar{\gamma}_\alpha^\beta e_\beta, \quad \gamma_\alpha^\gamma \bar{\gamma}_\gamma^\beta = \delta_\alpha^\beta. \quad (3.10)$$

In this case the second double $\tilde{H}(\mathcal{A})$ can be realized through the first one $H(\mathcal{A})$:

$$\tilde{e}_\alpha = \gamma_\alpha^\beta \bar{e}_\beta, \quad \tilde{e}^\alpha = \bar{\gamma}_\beta^\alpha \bar{e}^\beta, \quad (3.11)$$

where the overline means the opposite multiplication, which can be realized by the transposition operation. The four different S -matrices can be expressed now only in terms of the original one:

$$\tilde{S} = S^t, \quad S' = (S^{-1})^{t_1}, \quad S'' = (S^{t_2})^{-1}, \quad (3.12)$$

where t and t_i mean the full and partial transpositions respectively. Clearly, all the pentagon relations (3.2) and (3.8) are reduced to (1.2). Thus, formula (3.5) enables to construct solutions to the Yang-Baxter relation from invertible and cross-invertible¹ solutions to the pentagon relation (1.2).

The above construction can be generalized also to the non-constant case. For this consider some invertible and cross-invertible solution to the non-constant pentagon relation (see [8] for the definition, and [7], for the example) which can be written in the following form:

$$\begin{aligned} & S_{12}(z_0, z_1, z_2, z_3) S_{13}(z_0, z_1, z_3, z_4) S_{23}(z_1, z_2, z_3, z_4) \\ &= S_{23}(z_0, z_2, z_3, z_4) S_{12}(z_0, z_1, z_2, z_4), \end{aligned} \quad (3.13)$$

where z_0, \dots, z_4 are some parameters of any nature. Substituting non-constant S 's into (3.5), we obtain an R -matrix, depending of sixteen parameters:

$$\begin{aligned} R_{12,34}(\hat{z}) &= (S_{14}^{t_4}(z_{11}, \dots, z_{14}))^{-1} S_{13}(z_{21}, \dots, z_{24}) \\ &\times S_{24}^t(z_{31}, \dots, z_{34}) (S_{23}^{-1}(z_{41}, \dots, z_{44}))^{t_2}, \end{aligned} \quad (3.14)$$

where \hat{z} is a 4-by-4 matrix with entries z_{ij} for $i, j = 1, \dots, 4$. Substituting six such R -matrices with different arguments into (3.7), we get the following equation:

$$R_{11',22'}(\hat{x}) R_{11',33'}(\hat{y}) R_{22',33'}(\hat{z}) = R_{22',33'}(\hat{z}') R_{11',33'}(\hat{y}') R_{11',22'}(\hat{x}'), \quad (3.15)$$

which is satisfied provided the matrices \hat{x}, \dots, \hat{x}' are constrained in the special way:

$$\begin{aligned} x_{23} &= x_{43} = z_{12} = z_{22} = x'_{23} = x'_{43} = z'_{12} = z'_{22}, \\ x_{13} &= x_{33} = z_{32} = z_{42} = x'_{13} = x'_{33} = z'_{32} = z'_{42}, \end{aligned}$$

¹ By cross-invertibility we mean the invertibility of the partially transposed matrix S^{t_2}

$$x_{22} = x_{42} = x'_{22} = x'_{42} = y_{i2}, \quad x_{12} = x_{32} = x'_{12} = x'_{32} = y'_{i2}, \\ z_{13} = z_{23} = z'_{13} = z'_{23} = y_{i3}, \quad z_{33} = z_{43} = z'_{33} = z'_{43} = y'_{i3}, \quad i = 1, \dots, 4,$$

$$x_{11} = x'_{11} = y'_{11} = y'_{21} = z'_{31} = z'_{41}, \quad z_{14} = z'_{14} = x'_{24} = x'_{44} = y_{24} = y_{44}, \\ x_{21} = x'_{21} = y_{11} = y_{21} = z'_{11} = z'_{21}, \quad z_{24} = z'_{24} = x_{24} = x_{44} = y_{14} = y_{34}, \\ x_{31} = x'_{31} = y'_{31} = y'_{41} = z_{31} = z_{41}, \quad z_{34} = z'_{34} = x_{14} = x_{24} = y'_{14} = y'_{34}, \\ x_{41} = x'_{41} = y_{31} = y_{41} = z_{11} = z_{21}, \quad z_{44} = z'_{44} = x'_{14} = x'_{24} = y'_{24} = y'_{44}. \quad (3.16)$$

Apparently, there should exist a reparametrization, simplifying these constraints, and admitting some “rapidity” picture.

To conclude the section note, that formula (3.5) has a typical “box” structure, used for the construction of solutions to the Yang-Baxter equation in terms of those to the “twisted” Yang-Baxter equations [6].

4 Examples

Consider several examples, realizing the general constructions of Section 2.

For a given finite group \mathcal{G} consider its group algebra as a Hopf algebra \mathcal{A} . The multiplication relations of the corresponding Heisenberg double are as follows:

$$e_g e_h = e_{gh}, \quad e^g e^h = \delta_{g,h} e^g, \quad e^g e_h = e_h e^{gh}, \quad g, h \in \mathcal{G}. \quad (4.1)$$

This example generates the infinite sequence of finite dimensional solutions to the pentagon relation (1.2). Particularly, in the adjoint representation (2.3), the matrix elements of the canonical element read:

$$\langle g, h | S | g', h' \rangle = \delta_{gh}^{g'} \delta_h^{h'}. \quad (4.2)$$

Let Hopf algebra \mathcal{A} be now the algebra $\mathbf{C}[x]$ with the co-multiplication

$$\Delta(x) = x \otimes 1 + 1 \otimes x. \quad (4.3)$$

For a linear base take the normalized monomials:

$$e_m = x^m / m!, \quad m = 0, 1, \dots \quad (4.4)$$

The dual algebra \mathcal{A}^* is also the algebra $\mathbf{C}[\bar{x}]$ with the co-multiplication of the form (4.3), the dual base being

$$e^m = \bar{x}^m, \quad m = 0, 1, \dots \quad (4.5)$$

The Heisenberg double $H(\mathcal{A})$ is defined by the Heisenberg permutation relation between x and \bar{x} :

$$x\bar{x} - \bar{x}x = 1, \quad (4.6)$$

and the canonical element is just the usual exponent:

$$S = \sum_{m=0}^{\infty} e_m \otimes e^m = \exp(x \otimes \bar{x}). \quad (4.7)$$

The pentagon relation (1.2) is an evident consequence of the permutation relation (4.6).

The next example is less trivial. Let algebra \mathcal{A} be a deformed universal enveloping algebra of the Lie algebra:

$$HE - EH = E, \quad (4.8)$$

with the co-multiplications [3]:

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(E) = E \otimes \exp(hH) + 1 \otimes E, \quad (4.9)$$

where the complex parameter h is chosen to have a positive real part. This algebra coincides with Borel subalgebra of $U_q(sl(2))$ where $q = \exp(-h)$. For a linear base take again normalized monomials:

$$e_{m,n} = H^m E^n / m!(q)_n, \quad m, n = 0, 1, \dots, \quad (4.10)$$

where

$$(q)_n = \begin{cases} 1 & n = 0; \\ (1 - q) \dots (1 - q^n) & n > 0. \end{cases} \quad (4.11)$$

The dual algebra is generated also by two elements \overline{H} and F with the permutation relation:

$$\overline{H}F - F\overline{H} = -hF, \quad (4.12)$$

and co-multiplications:

$$\Delta(\overline{H}) = \overline{H} \otimes 1 + 1 \otimes \overline{H}, \quad \Delta(F) = F \otimes \exp(-\overline{H}) + 1 \otimes F, \quad (4.13)$$

the dual base being

$$e^{m,n} = \overline{H}^m F^n, \quad m, n = 0, 1, \dots \quad (4.14)$$

The permutation relations of the Heisenberg double $H(\mathcal{A})$ read

$$H\overline{H} - \overline{H}H = 1, \quad E\overline{H} = \overline{H}E, \quad (4.15)$$

$$HF - FH = -F, \quad EF - FE = (1 - q)q^{-H}. \quad (4.16)$$

The canonical element is given by the formula:

$$S = \sum_{m,n=0}^{\infty} e_{m,n} \otimes e^{m,n} = \exp(H \otimes \overline{H})(E \otimes F; q)_{\infty}^{-1}, \quad (4.17)$$

where

$$(x; q)_{\infty} = (1 - x)(1 - xq) \dots \quad (4.18)$$

The pentagon relation (1.2) for the element (4.17) can be rewritten by the use of above permutation relations in the following form:

$$(U; q)_{\infty}([U, V]/(1 - q); q)_{\infty}(V; q)_{\infty} = (V; q)_{\infty}(U; q)_{\infty}, \quad (4.19)$$

where the square brackets denote the commutator, and

$$U = E_2 F_3, \quad V = E_1 F_2. \quad (4.20)$$

Operators U and V satisfy the following algebraic relations:

$$W \equiv UV - qVU, \quad [U, W] = [V, W] = 0, \quad (4.21)$$

which mean that the element W lies in the center of the algebra, generated by operators U and V . In particular case, where $W = 0$, formula (4.19) coincides with the quantum dilogarithm identity of [4]. The generalized form (4.19) of the latter has been found earlier by Volkov [11].

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